Another way to find the Euer-Lagrange equations system in Classical Mechanics: The Q-Gradient.

(A proposal based entirely on the Newton's 2nd law) Let be $\overrightarrow{P} = \sum_k \overrightarrow{p}_k$,

$$\frac{d\overrightarrow{P}}{dt} = \overrightarrow{F}_{net}$$
$$\frac{d\overrightarrow{P}}{dt} = \overrightarrow{F}_{external} + \overrightarrow{F}_{PairBy3rdLaw}$$
$$\frac{d\overrightarrow{P}}{dt} = \overrightarrow{F}_{external} + \cancel{\overrightarrow{F}_{PairBy3rdLaw}}$$
$$\frac{d\overrightarrow{P}}{dt} - \overrightarrow{F}_{ext} = \overrightarrow{0}$$
$$\frac{d}{dt}\left(\overrightarrow{P}\right) - \overrightarrow{F}_{ext} = \overrightarrow{0}$$
$$\left\{\frac{d}{dt}\left(\overrightarrow{P}\right) - \overrightarrow{F}_{ext}\right\} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r} = \overrightarrow{0} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r}$$
$$\frac{d}{dt}\left(\overrightarrow{P}\right) \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r} - \left(\overrightarrow{F}_{ext} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r}\right) = \overrightarrow{0}$$

Since this is not the MORE-GENERAL-POSSIBLE situation about such time derivative, we guess that the following is the right choice:

$$\frac{d}{dt}\left(\overrightarrow{P} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r}\right) - \left(\overrightarrow{F}_{ext} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r}\right) = \overrightarrow{0}$$
$$\frac{d\overrightarrow{P}}{dt} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r} + \overrightarrow{P} \cdot \overrightarrow{\nabla}_Q \sum \frac{d\overrightarrow{r}}{dt} - \left(\overrightarrow{F}_{ext} \cdot \overrightarrow{\nabla}_Q \sum \overrightarrow{r}\right) = \overrightarrow{0}$$

Now, by a careful application of this vector equation, we should be able to find the Euler-Lagrange system of equations for any physical system already tractable in Classical Mechanics using the Lagrangian formalism. In fact, at this point we have proved the following identity: (γ is a constant)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) = \gamma \frac{d}{dt}\left(\overrightarrow{P} \cdot \frac{\partial}{\partial Q_j} \sum \overrightarrow{r}\right)$$

But wait. What is meant by $\overrightarrow{\nabla}_Q$ as operator?... If the physical system which we are treating have S degrees of freedom, then $\overrightarrow{\nabla}_Q$ is a first-order operator defined in a cartesian space (S-dimentional) as:

$$\overrightarrow{\nabla}_Q = \sum_{i=1}^{S} \frac{\partial}{\partial Q_i} \overrightarrow{e}_i$$

Example. Disipative forces: Air resistance

$$\dot{\vec{r}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\left|\dot{\vec{r}}\right| = \sqrt{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\left|\vec{F}_{air}\right| = \beta \left|\dot{\vec{r}}\right|^n$$

$$\vec{F}_{air} \cdot \dot{\vec{r}} = \left|\vec{F}_{air}\right| \left|\dot{\vec{r}}\right| \cos(\pi) = -\beta \left|\dot{\vec{r}}\right|^{(n+1)}$$

$$\frac{\left(\vec{F}_{air} \cdot \dot{\vec{r}}\right)}{\left|\dot{\vec{r}}\right|} \frac{\dot{\vec{r}}}{\left|\dot{\vec{r}}\right|} = -\beta \left|\dot{\vec{r}}\right|^{(n+1)} \frac{\dot{\vec{r}}}{\left|\dot{\vec{r}}\right|^2} = -\beta \left|\dot{\vec{r}}\right|^{(n-1)} \dot{\vec{r}}$$

Proportionality with the n-th power of the velocity: $(n > 0)$

$$\vec{F}_{air} = -\beta \left(\dot{x}^2 + \dot{y}^2\right)^{\frac{(n-1)}{2}} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

Case $n = 1$. Linear resistance:

$$\vec{F}_{air} = -\beta \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\sum \vec{F}_{External} = \vec{F}_{weight} + \vec{F}_{air} = -mg \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \beta \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\sum \vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\partial_x \Sigma \vec{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\partial_y \Sigma \vec{r} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\partial_x \Sigma \dot{\vec{r}} = \partial_y \Sigma \dot{\vec{r}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{P} = m \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\dot{\vec{P}} \cdot \vec{\nabla} \Sigma \vec{r} + \vec{P} \cdot \vec{\nabla} \Sigma \dot{\vec{r}} - \vec{F}_{Ext.} \cdot \vec{\nabla} \Sigma \vec{r} = \vec{0}$$

$$\begin{pmatrix} \ddot{x} + \frac{\beta}{m}\dot{x} \\ \ddot{y} + g + \frac{\beta}{m}\dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$